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A Study On The Hyperbola

$$y^2 = 11x^2 + 1$$

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Abstract

The binary quadratic equation $y^2 = 11x^2 + 1$ is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

Keywords: Binary quadratic, hyperbola, integral solutions, pell equation.

Introduction

Any non-homogeneous binary quadratic equation of the form $y^2 - Dx^2 = 1$,where D is a given positive non-square integer, requiring integer solutions for x and y is called Pellian equation (also known pell-Fermat equation). In cartesian co-ordinates, the equation has the form of a hyperbola. The pellian equation has infinitely many distinct integer solutions as long as D is not a perfect square and the solutions are easily generated recursively from a single fundamental solution, namely, the solution with x, y positive integers of smallest possible size. One may refer [1-9] for a few choices of Pellian equations along with their corresponding integer solutions.

The solutions to Pellian equations have long been of interest to mathematicians. Even small values of D can lead to fundamental solutions which are quite large. For example, when D=61, the fundamental solution is (1766319049, 226153980). The above results motivated us to search for integer solutions to other choices of Pellian equation. This paper concerns with the Pellian equation $y^2 = 11x^2 + 1$, a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

Method of analysis:

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$y^2 = 11x^2 + 1 \quad (1)$$

The smallest positive integer solution is $x_0 = 3, y_0 = 10$

If (x_n, y_n) represents the general solution of (1), then

$$x_n = \left(1/2\sqrt{D}\right) g_n \quad (2)$$

$$y_n = (1/2) f_n \quad (3)$$

where

$$f_n = \left(10 + 3\sqrt{11}\right)^{n+1} + \left(10 - 3\sqrt{11}\right)^{n+1}$$
$$g_n = \left(10 + 3\sqrt{11}\right)^{n+1} - \left(10 - 3\sqrt{11}\right)^{n+1}$$

A few numerical solutions to (1) are presented in table below:

Table: Numerical solutions

N	x_n	y_n
0	3	10
1	60	199
2	1197	3970
3	23880	79201
4	476403	1580050
5	9504180	31521799

Observations:

- $x_{2n-1} \equiv 0(\bmod 10), n = 1,2,3.......$
- $y_{2n} \equiv 0(\bmod 10), n = 0,1,2........$
- $x_n \equiv 0(\bmod 3), n = 0,1,2........$
- A few interesting relations among the solutions are given below:

  - $33x_n - y_{n+1} + 10y_n = 0$
  - $33x_{n+1} - 10y_{n+1} + y_n = 0$
  - $33x_{n+2} - 199y_{n+1} + 10y_n = 0$
  - $660x_n - y_{n+2} + 199y_n = 0$
  - $660x_{n+1} - 10y_{n+2} + 10y_n = 0$
  - $660x_{n+2} - 199y_{n+2} + y_n = 0$
  - $33x_n - 10y_{n+2} + 199y_{n+1} = 0$
  - $33x_{n+1} - y_{n+2} + 10y_{n+1} = 0$
  - $33x_{n+2} - 10y_{n+2} + y_{n+1} = 0$
  - $3y_n - x_{n+1} + 10x_n = 0$
  - $3y_{n+1} - 10x_{n+1} + x_n = 0$
  - $3y_{n+2} - 199x_{n+1} + 10x_n = 0$
  - $3y_n - 10x_{n+2} + 199x_n = 0$
  - $3y_{n+2} - 10x_{n+2} + x_{n+1} = 0$
  - $60y_n - x_{n+2} + 199x_n = 0$
  - $60y_{n+1} - 199x_{n+2} + x_n = 0$
  - $60y_{n+2} - 10x_{n+2} + 10x_n = 0$

- Expressions representing square integers:

  - $2y_{2n+1} + 2$
  - $2(20y_{2n+2} - y_{2n+3}) + 2$

- $\frac{1}{5}(y_{2n+2}-33x_{2n+1})+2$
- $2(10y_{2n+2}-33x_{2n+2})+2$
- $\frac{1}{5}(199y_{2n+2}-33x_{2n+3})+2$
- $\frac{2}{199}(y_{2n+3}-660x_{2n+1})+2$
- $2(y_{2n+3}-66x_{2n+1})+2$
- $2(y_{2n+3}-660x_{2n+3})+2$
- $\frac{2}{3}(x_{2n+2}-10x_{2n+1})+2$
- $\frac{1}{30}(x_{2n+3}-199x_{2n+1})+2$
- $\frac{2}{3}(10x_{2n+3}-199x_{2n+2})+2$

➢ Expressions representing cubical integers:

- $2y_{3n+2}+6y_n$
- $2(20y_{3n+3}-y_{3n+5})+3f_n$
- $\frac{1}{5}(y_{3n+3}-33x_{3n+2})+3f_n$
- $2(10y_{3n+3}-33x_{3n+3})+3f_n$
- $\frac{1}{5}(199y_{3n+3}-33x_{3n+5})+3f_n$
- $\frac{2}{199}(y_{3n+4}-660x_{3n+2})+3f_n$
- $2(y_{3n+4}-66x_{3n+3})+3f_n$
- $2(199y_{3n+4}-660x_{3n+4})+3f_n$
- $\frac{2}{3}(x_{3n+3}-10x_{3n+2})+3f_n$

- $\frac{1}{30}(x_{3n+4}-199x_{3n+2})+3f_n$
- $\frac{2}{3}(10x_{3n+4}-199x_{3n+3})+3f_n$

➢ Expressions representing biquadratic integers:

- $2y_{4n+3}+4(4y_n{}^2)-2$
- $2(20y_{4n+4}-y_{4n+5})+16y_n{}^2-2$
- $\frac{1}{5}(y_{4n+4}-33x_{4n+3})+16y_n{}^2-2$
- $2(10y_{4n+4}-33y_{4n+4})+16y_n{}^2-2$
- $\frac{1}{5}(199y_{4n+4}-33x_{4n+5})+16y_n{}^2-2$
- $\frac{2}{199}(y_{4n+5}-660x_{4n+3})+16y_n{}^2-2$
- $2(y_{4n+5}-66x_{4n+4})+16y_n{}^2-2$
- $2(199y_{4n+5}-660x_{4n+5})+16y_n{}^2-2$
- $\frac{2}{3}(x_{4n+4}-10x_{4n+3})+16y_n{}^2-2$
- $\frac{1}{30}(x_{4n+5}-199y_{4n+3})+16y_n{}^2-2$
- $\frac{2}{3}(10y_{4n+5}-x_{4n+4})+16y_n{}^2-2$

➢ Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

Choise1: Let $X = y_n$, $Y = y_{n+1}-10y_n$

$Y^2 = 99X^2 - 99$

Note that (X, Y) satisfies the hyperbola

Choise2: Let $X = y_n$, $Y = y_{n+2} - 199y_n$

$Y^2 = 39600X^2 - 39600$

Note that (X, Y) satisfies the hyperbola

Choise3: Let $X = y_n$, $Y = x_n$

$11Y^2 = X^2 - 1$

Note that (X, Y) satisfies the hyperbola

Choise4: Let $X = y_n$, $Y = x_{n+1} - 3y_n$

$11Y^2 = 100X^2 - 100$

Note that (X, Y) satisfies the hyperbola

Choise5: Let $X = y_n$, $Y = x_{n+2} - 60y_n$

$11Y^2 = 39601X^2 - 39601$

Note that (X, Y) satisfies the hyperbola

Choise6: Let $X = 20y_{n+1} - y_{n+2}$, $Y = 10y_{n+2} - 199y_{n+1}$

$Y^2 = 99X^2 - 99$

Note that (X, Y) satisfies the hyperbola

Choise7: Let $X = y_{n+1} - 33x_n$, $Y = x_n$

$1100Y^2 = X^2 - 100$

Note that (X, Y) satisfies the hyperbola

Choise8: Let $X = 10y_{n+1} - 33x_{n+1}$, $Y = 10x_{n+1} - 3y_{n+1}$

$11Y^2 = X^2 - 1$

Note that (X, Y) satisfies the hyperbola

Choise9: Let $X = 199y_{n+1} - 33x_{n+2}$, $Y = x_{n+2} - 6y_{n+1}$

$1100Y^2 = X^2 - 100$

Note that (X, Y) satisfies the hyperbola

Choise10: Let $X = y_{n+2} - 660x_n$, $Y = x_n$

$435611Y^2 = X^2 - 39601$

Note that (X, Y) satisfies the hyperbola

Choise11: Let $X = y_{n+2} - 66x_{n+1}$, $Y = 199x_{n+1} - 3y_{n+2}$

$11Y^2 = 100X^2 - 100$

Note that (X, Y) satisfies the hyperbola

Choise12: Let $X = 199y_{n+2} - 660x_{n+2}$, $Y = 199x_{n+2} - 60y_{n+2}$

$11Y^2 = X^2 - 1$

Note that (X, Y) satisfies the hyperbola

Choise13: Let $X = x_{n+1} - 10x_n$, $Y = x_n$

$99Y^2 = X^2 - 9$

Note that (X, Y) satisfies the hyperbola

Choise14: Let $X = x_{n+2} - 199x_n$, $Y = x_n$

$39600Y^2 = X^2 - 3600$

Note that (X, Y) satisfies the hyperbola

Choise15: Let $X = 10x_{n+2} - 199x_{n+1}$, $Y = 20x_{n+1} - x_{n+2}$

$99Y^2 = X^2 - 9$

Note that (X, Y) satisfies the hyperbola

- Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas

Choise16: Let $X = y_{2n+1} + 1$, $Y = y_{n+1} - 10y_n$

$2Y^2 = 99X - 198$

Note that (X, Y) satisfies the parabola.

Choise17: Let $X = y_{2n+1} + 1$, $Y = y_{n+2} - 199y_n$

$Y^2 = 19800X - 39600$

Note that (X, Y) satisfies the parabola.

Choise18: Let $X = y_{2n+1} + 1$, $Y = x_n$

$22Y^2 = X - 2$

Note that (X, Y) satisfies the parabola.

Choise19: Let $X = y_{2n+1} + 1$, $Y = x_{n+1} - 3y_n$

$11Y^2 = 50X - 100$

Note that (X, Y) satisfies the parabola.

Choise20: Let $X = y_{2n+1} + 1$, $Y = x_{n+2} - 60y_n$

$22Y^2 = 39601X - 79202$

Note that (X, Y) satisfies the parabola.

Choise21: Let $X = 20y_{2n+2} - y_{2n+3}$, $Y = 10y_{n+2} - 199y_{n+1}$

$2Y^2 = 99X - 99$

Note that (X, Y) satisfies the parabola.

Choise22: Let $X = y_{2n+2} - 33x_{2n+1}$, $Y = x_n$

$220Y^2 = X - 10$

Note that (X, Y) satisfies the parabola.

Choise23: Let $X = 10y_{2n+2} - 33x_{2n+2}$, $Y = 10x_{n+1} - 3y_{n+1}$

$22Y^2 = X - 1$

Note that (X, Y) satisfies the parabola.

Choise24: Let $X = 199y_{2n+2} - 33x_{2n+3}$, $Y = x_{n+2} - 6y_{n+1}$

$220Y^2 = X - 10$

Note that (X, Y) satisfies the parabola.

Choise25: Let $X = y_{2n+3} - 660x_{2n+1},\ Y = x_n$

$$4378Y^2 = X - 199$$

Note that (X, Y) satisfies the parabola.

Choise26: Let $X = y_{2n+3} - 66x_{2n+2},\ Y = 199x_{n+1} - 3y_{n+2}$

$$11Y^2 = 50X - 50$$

Note that (X, Y) satisfies the parabola.

Choise27: Let $X = 199y_{2n+3} - 660x_{2n+3},\ Y = 199x_{n+2} - 60y_{n+2}$

$$22Y^2 = X - 1$$

Note that (X, Y) satisfies the parabola.

Choise28; Let $X = x_{2n+2} - 10x_{2n+1},\ Y = x_n$

$$66Y^2 = X - 3$$

Note that (X, Y) satisfies the parabola.

Choise29: Let $X = x_{2n+3} - 199x_{2n+1},\ Y = x_n$

$$1320Y^2 = X - 60$$

Note that (X, Y) satisfies the parabola.

Choise30: Let $X = 10x_{2n+3} - 199x_{2n+2},\ Y = 20x_{n+1} - x_{n+2}$

$$66Y^2 = X - 3$$

Note that (X, Y) satisfies the parabola.

- considering suitable values of X_n& Y_n, one generates 2^{nd} order Ramanujan numbers with base integers as real integers

For illustration, consider

$x_2 = 1197 = 1*1197 = 3*399 = 7*171 = 9*133 = 19*63 = 21*57\ (*)$

Now,

$1*1197 = 3*399$

$$\Rightarrow (1+1197)^2 + (3-399)^2 = (1-1197)^2 + (3+399)^2 = 1592020$$

$1*1197 = 7*171$

$$\Rightarrow (1+1197)^2 + (7-171)^2 = (1-1197)^2 + (7+171)^2 = 1462100$$

$1*1197 = 9*133$

$$\Rightarrow (1+1197)^2 + (9-133)^2 = (1-1197)^2 + (9+133)^2 = 1450580$$

$1*1197 = 19*63$

$$\Rightarrow (1+1197)^2 + (19-63)^2 = (1-1197)^2 + (19+63)^2 = 1437140$$

$1*1197 = 21*57$

$\Rightarrow (1+1197)^2 + (21-57)^2 = (1-1197)^2 + (25+57)^2 = 1436500$

$3*399 = 7*171$

$\Rightarrow (3+399)^2 + (7-171)^2 = (3-399)^2 + (7+171)^2 = 188500$

$3*399 = 9*133$

$\Rightarrow (3+399)^2 + (9-133)^2 = (3-399)^2 + (9+133)^2 = 176980$

$3*399 = 19*63$

$\Rightarrow (3+399)^2 + (19+63)^2 = (3-399)^2 + (19-63)^2 = 163540$

$3*399 = 21*57$

$\Rightarrow (3+399)^2 + (21-57)^2 = (3-399)^2 + (21+57)^2 = 162900$

$7*171 = 9*133$

$\Rightarrow (7+171)^2 + (9-133)^2 = (7-171)^2 + (9+133)^2 = 47060$

$7*171 = 19*63$

$\Rightarrow (7+171)^2 + (19-63)^2 = (7-171)^2 + (19+63)^2 = 33620$

$7*171 = 21*57$

$\Rightarrow (7+171)^2 + (21-57)^2 = (7-171)^2 + (21+57)^2 = 32980$

$9*133 = 19*63$

$\Rightarrow (9+133)^2 + (19-63)^2 = (9-133)^2 + (19+63)^2 = 22100$

$9*133 = 21*57$

$\Rightarrow (9+133)^2 + (21-57)^2 = (9-133)^2 + (21+57)^2 = 21460$

$19*63 = 21*57$

$\Rightarrow (19+63)^2 + (21-57)^2 = (19-63)^2 + (21+57)^2 = 8020$

Note:

$$1*1197 = 3*399$$
$$\Rightarrow 599^2 - 598^2 = 201^2 - 198^2$$
$$\Rightarrow 599^2 + 198^2 = 201^2 + 598^2$$
$$\Rightarrow 358801 + 39204 = 40401 + 357604 = 398005$$

$$1*1197 = 7*171$$
$$\Rightarrow 599^2 - 598^2 = 89^2 - 82^2$$
$$\Rightarrow 599^2 + 82^2 = 89^2 + 598^2$$
$$\Rightarrow 358801 + 6724 = 7921 + 357604 = 365525$$

$$1*1197 = 9*133$$
$$\Rightarrow 599^2 - 598^2 = 71^2 - 62^2$$
$$\Rightarrow 599^2 + 62^2 = 71^2 + 598^2$$
$$\Rightarrow 358801 + 3844 = 5041 + 357604 = 357604$$

$$1*1197 = 19*63$$
$$\Rightarrow 599^2 - 598^2 = 41^2 - 22^2$$
$$\Rightarrow 599^2 + 22^2 = 41^2 + 598^2$$
$$\Rightarrow 358801 + 484 = 1681 + 357604 = 359285$$
$$1*1197 = 21*57$$
$$\Rightarrow 599^2 - 598^2 = 39^2 - 18^2$$
$$\Rightarrow 599^2 + 38^2 = 18^2 + 598^2$$
$$\Rightarrow 358801 + 324 = 1521 + 357604 = 359125$$

$$3*399 = 7*171$$
$$\Rightarrow 201^2 - 198^2 = 89^2 - 82^2$$
$$\Rightarrow 201^2 + 82^2 = 89^2 + 198^2$$
$$\Rightarrow 40401 + 6724 = 7921 + 39204 = 47125$$

$$3*399 = 9*133$$
$$\Rightarrow 201^2 - 198^2 = 71^2 - 62^2$$
$$\Rightarrow 201^2 + 62^2 = 71^2 + 198^2$$
$$\Rightarrow 40401 + 3844 = 5041 + 39204 = 44245$$

$3*399=19*63$

$\Rightarrow 201^2-198^2=41^2-22^2$

$\Rightarrow 201^2+22^2=41^2+198^2$

$\Rightarrow 40401+404=1681+39204=40885$

$3*399=21*57$

$\Rightarrow 201^2-198^2=39^2-18^2$

$\Rightarrow 201^2+18^2=39^2+198^2$

$\Rightarrow 40401+324=1521+39204=40725$

$7*171=9*133$

$\Rightarrow 89^2-82^2=71^2-62^2$

$\Rightarrow 89^2+62^2=71^2+82^2$

$\Rightarrow 7921+3844=5041+6724=11765$

$7*171=19*63$

$\Rightarrow 89^2-82^2=41^2-22^2$

$\Rightarrow 89^2+22^2=41^2+82^2$

$\Rightarrow 7921+484=1681+6724=8405$

$7*171=21*57$

$\Rightarrow 89^2-82^2=39^2-18^2$

$\Rightarrow 89^2+18^2=39^2+82^2$

$\Rightarrow 7921+324=1521+6724=8245$

$9*133=19*63$

$\Rightarrow 71^2-62^2=41^2-22^2$

$\Rightarrow 71^2+22^2=41^2+62^2$

$\Rightarrow 5041+484=1681+3844=5525$

$9*133=21*57$

$\Rightarrow 71^2-62^2=39^2-18^2$

$\Rightarrow 71^2+18^2=39^2+62^2$

$\Rightarrow 5041+324=1521+3844=5365$

$19*63=21*57$

$\Rightarrow 41^2-22^2=39^2-18^2$

$\Rightarrow 41^2+18^2=39^2+22^2$

$\Rightarrow 1681+324=1521+484=2005$

Thus **398005, 365525, 362645, 359285, 359125, 47125, 44245, 40845, 40725, 11765, 80405, 80245, 5525, 5365, 2005, 1592020, 1462100, 1450580, 1437140, 1436500, 188500, 176980, 163540, 162900, 47060, 33620, 32980, 22100, 21420, 8020** represent 2^{nd} order Ramanujan numbers.

- Considering suitable values of x_n & y_n, one generates 2^{nd} order Ramanujan numbers with base integers as Gaussian integers.

For illustration, consider again x_2 represented by (*),
Now,
$1*1197=3*399$

$\Rightarrow (i+1197)^2+(3i-399)^2=(i-1197)^2+(3+399)^2=1592000$

$1*1197=7*171$

$\Rightarrow (i+1197)^2+(7i-171)^2=(i-1197)^2+(7i+171)^2=1462000$

$1*1197=9*133$

$\Rightarrow (i+1197)^2+(9i-133)^2=(i-1197)^2+(9i+133)^2=1450416$

$1*1197=19*63$

$\Rightarrow (i+1197)^2+(19i-63)^2=(i-1197)^2+(19i+63)^2=1436416$

$1*1197=21*57$

$\Rightarrow (i+1197)^2+(21i-57)^2=(i-1197)^2+(21i+57)^2=1435616$

$3*399=7*171$

$\Rightarrow (3i+399)^2+(7i-171)^2=(3i-399)^2+(7i+171)^2=188384$

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$3*399 = 9*133$

$$\Rightarrow (3i+399)^2 + (9i-133)^2 = (3i-399)^2 + (9i+133)^2 = 176800$$

$3*399 = 19*63$

$$\Rightarrow (3i+399)^2 + (19i+63)^2 = (3i-399)^2 + (19i-63)^2 = 19519$$

$3*399 = 21*57$

$$\Rightarrow (3i+399)^2 + (21i-57)^2 = (3i-399)^2 + (21i+57)^2 = 162000$$

$7*171 = 9*133$

$$\Rightarrow (7i+171)^2 + (9i-133)^2 = (7i-171)^2 + (9i+133)^2 = 46800$$

$7*171 = 19*63$

$$\Rightarrow (7i+171)^2 + (19i-63)^2 = (7i-171)^2 + (19i+63)^2 = 32800$$

$7*171 = 21*57$

$$\Rightarrow (7i+171)^2 + (21i-57)^2 = (7i-171)^2 + (21i+57)^2 = 32000$$

$9*133 = 19*63$

$$\Rightarrow (9i+133)^2 + (19i-63)^2 = (9i-133)^2 + (19i+63)^2 = 21216$$

$9*133 = 21*57$

$$\Rightarrow (9i+133)^2 + (21i-57)^2 = (9i-133)^2 + (21i+57)^2 = 20416$$

$19*63 = 21*57$

$$\Rightarrow (19i+63)^2 + (21i-57)^2 = (19i-63)^2 + (21i+57)^2 = 6416$$

Note that **1592000, 1462000, 1450416, 1436416, 1435616, 188384, 176800, 19519, 162000, 46800, 32800, 32000, 21216, 20416, 6416** represent 2^{nd} order Ramanujan numbers with base integers as gaussian integers.

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